

Second Order Adjacency Energy of Some Graphs

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Abstract

Graph energy is well known graph invariant, defined as the sum of absolute values of the eigenvalues of the adjacency matrix of a graph. Motivated by the study of distance based spectral matrices, the second order adjacency matrix is defined whose (i,j) -th entry is equal to one if the distance between the vertices v_i and v_j is two and zero otherwise. The second order adjacency energy for certain classes of graph such as path, cycle, star and complete graph were computed.

Keywords: Adjacency matrix, Eigenvalues, Energy of graph, Second order adjacency energy.

1. Introduction

In 1978, Gutman [2],[3] introduced the fundamental concept of graph energy using the adjacency matrix, which laid the foundation for spectral energy studies in graph theory. In 2008, G. Indulal, I. Gutman and A. Vijayakumar investigated the distance energy of graphs based on the distance matrix and analyzed its properties [10].

More recently, In 2021, Wang et al.[2, 7] provided a comprehensive survey on distance-related energy matrices, summarizing various distance-based spectral indices and their applications. Research on graph energy focused primarily on the adjacency matrix [1] after that this energy expanded to other matrix representations such as the Laplacian matrix, signless Laplacian matrix, distance matrix and eccentricity matrix [6].

The adjacency matrix provides an intermediate representation between local adjacency information and global distance-based measures. Unlike the adjacency matrix, which records direct connections between vertices, the second order adjacency matrix captures whether two vertices are at distance exactly two.

Such indirect interactions through intermediate vertices play an important role in applications such as social networks, communication systems and molecular structures, where indirect influence can be as significant as direct connections [8].

In many applications-such as chemical graph theory, communication networks, biological networks and social interaction-vertices that are not directly adjacent may still influence each other through a common neighbour. These interactions correspond to paths of length two and are not reflected in the classical adjacency matrix [4,5]. To capture such indirect interactions, it is natural to extend the notion of adjacency beyond immediate neighbors.

In this paper, finding the second order adjacency energy for fundamental graph classes such as path, cycle, complete and star graphs. These graphs serve as canonical models due to their simple structure and well-understood spectral properties [1],[8].

2. Preliminaries

In this section, basic definitions and notations from spectral graph theory which are given by

Let $G = (V(G), E(G))$ be a simple, finite, undirected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The order and size of a graph are denoted by $n = |V(G)|$ and $m = |E(G)|$, respectively.

Definition 2.1. Let $G = (V, E)$ be a connected graph and let $u, v \in V(G)$. The distance between the vertices u and v is denoted by $d(u, v)$, and also which is defined as the length of the shortest path connecting u and v .

Definition 2.2. A path graph P_n is a connected graph on n vertices in which exactly two vertices have degree 1 and all remaining vertices have degree 2.

Definition 2.3. A cycle graph C_n is a simple graph on with at least 3 vertices in which every vertex has degree 2.

Definition 2.4. A star graph S_n is a tree on n vertices with one vertex of degree $n - 1$ and the remaining $n - 1$ vertices having degree 1.

Definition 2.5. A complete graph is an undirected graph in which every pair of distinct vertices is connected by a unique edge. The complete graph on n vertices is denoted by K_n .

Definition 2.6. For an $n \times n$ matrix A and a non-zero vector u , if multiplying A by u simply scales u by a factor λ , where λ is a scalar, then u is called an eigenvectors of A and λ is the corresponding eigenvalue such that $Au = \lambda u$.

Definition 2.7. The spectrum of a graph G with respect to its adjacency matrix $A(G)$ is the multiset of eigenvalues of $A(G)$ and is denoted by

$\text{Spec}(A(G)) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

Definition 2.8. The adjacency matrix of a graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ is denoted by $A(G) = [a_{ij}]$ and is an $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.9. The distance matrix of G is the $n \times n$ matrix

$$D(G) = (d_{ij}), \quad \text{where } d_{ij} = d(v_i, v_j),$$

where $d(v_i, v_j)$ denotes the distance between vertices v_i and v_j , such as distance is the length of shortest path from v_i to v_j .

Definition 2.10. Let G be a graph of order n and let A be the adjacency matrix of G . The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A are called the eigenvalues of G . The energy of a graph G is given by

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

Definition 2.11. Let P_n be the path graph on n vertices. The adjacency eigenvalues of P_n are

$$\lambda_k = 2 \cos \left(\frac{k\pi}{n+1} \right), \quad k = 1, 2, \dots, n.$$

The energy of the path graph P_n is

$$E(P_n) = \sum_{k=1}^n \left| 2 \cos \left(\frac{k\pi}{n+1} \right) \right|.$$

Definition 2.12. The energy of the complete graph K_n is given by

$$E(K_n) = 2(n-1)$$

where the adjacency eigenvalues of K_n are $n-1$ and -1 with multiplicity $n-1$.

Definition 2.13. Let S_n be a star graph and then the energy of S_n is given by

$$E(S_n) = 2\sqrt{n-1}$$

where $\lambda_k(S_n)$ are $\sqrt{n-1}, -\sqrt{n-1}, 0$.

3. Second order Adjacency Energy

Let $G = (V, E)$ be a simple undirected graph of order n with vertex set $V(G)$ and edge set $E(G)$. Then the second order adjacency matrix of G is an $n \times n$ matrix $A^{(2)}(G) = [a_{ij}^{(2)}]$ defined as follows :

$$a_{ij}^{(2)} = \begin{cases} 1, & \text{if } d(v_i, v_j) = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Here $d(v_i, v_j)$ is defined as the distance between two vertices v_i and v_j .

Let $\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)}$ are the eigenvalues of $A^{(2)}(G)$. Then the energy $E^{(2)}(G)$ is defined as

$$E^{(2)}(G) = \sum_{i=1}^n |\lambda_i^{(2)}|$$

3.1 Example of Second Order Adjacency Energy

Example 1. Let G be a graph and the vertex set is

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}.$$

The pairs of vertices at distance two are $(v_1, v_3), (v_2, v_4), (v_3, v_5)$

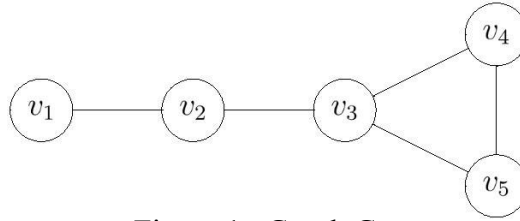


Figure 1 : Graph G

Hence, the second order adjacency matrix of G is

$$A^{(2)}(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalue of $A^{(2)}(G)$ are $\lambda = \{\sqrt{2}, -\sqrt{2}, 0, 0, 0\}$

Therefore, the second order adjacency energy of G is

$$E^{(2)}(G) = \sum_{i=1}^n |\lambda_i^{(2)}|.$$

$$E^{(2)}(G) = |\sqrt{2}| + |-\sqrt{2}| = 2\sqrt{2}.$$

$$E^{(2)}(G) = 2\sqrt{2} = 2.828.$$

4. Second Order Adjacency Energy of Some families of Graphs

Theorem 4.1. Let P_n be the path graph on $n \geq 3$ vertices with $V(G)$ be the vertex set and $E(G)$ be the edge set of G . Then the second order adjacency energy is defined as

$$E^{(2)}(P_n) = \sum_{k=1}^n \left| \lambda_k^{(2)} \right|, \quad n \geq 3.$$

The eigenvalues of the path graph $\lambda_k^{(2)}$ are defined as,

When n is even ($n = 2m$), then P_n decomposes into two independent paths of P_m and P_m , where m is an integer.

$$\lambda_k^{(2)} = 2 \cos \left(\frac{k\pi}{m+1} \right), \quad k = 1, 2, \dots, m.$$

Here each eigenvalue is exist with multiplicity 2.

When n is odd ($n = 2m + 1$), then P_n decomposes into P_{m+1} and P_m give eigenvalues

$$\lambda_k^{(2)} = \begin{cases} 2 \cos \left(\frac{k\pi}{m+2} \right), & k = 1, 2, \dots, m+1 \\ 2 \cos \left(\frac{k\pi}{m+1} \right), & k = 1, 2, \dots, m. \end{cases}$$

Proof. Let P_n be the path graph on $n \geq 3$ with $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G and $E(G)$ be the edge set $E(G) = \{e_1, e_2, \dots, e_n\}$ for the path graph P_n , the second order adjacency matrix $A^{(2)}(G) = (a_{ij}^{(2)})$ where

$$a_{ij}^{(2)} = \begin{cases} 1, & \text{if } d(v_i, v_j) = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the matrix is,

$$A^{(2)}(P_n) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

This is a symmetric band matrix of bandwidth 2.

Let the standard adjacency matrix $A^{(2)}(P_n)$ be,

$$A^{(2)}(P_n)x_n = \lambda x_n. \tag{1}$$

In P_n , the vertices at distance two from a vertex v_n are v_{n-2} and v_{n+2} whenever they exist. For interior vertices,

$$x_{n-2} + x_{n+2} = \lambda x_n, \quad n \geq 3. \quad (2)$$

Assume,

$$x_n = \sin n\theta. \quad (3)$$

$$\sin(n-2)\theta + \sin(n+2)\theta = \lambda \sin n\theta. \quad (4)$$

Consider the left side of equation (4),

$$\begin{aligned} \sin(n-2)\theta + \sin(n+2)\theta &= 2 \cos \left(\frac{(n-2) - (n+2)}{2} \theta \right) \sin \left(\frac{(n-2) + (n+2)}{2} \theta \right) \\ &= 2 \cos 2\theta \sin n\theta \end{aligned}$$

From (4),

$$\begin{aligned} 2 \cos 2\theta \sin n\theta &= \lambda \sin n\theta \\ \lambda &= 2 \cos 2\theta \end{aligned} \quad (5)$$

Since the boundary conditions are

$$\begin{aligned} x_0 &= x_{n+2} = 0. \\ x_{n+2} &= \sin(n+2)\theta = 0 \\ (n+2)\theta &= \sin^{-1}(0) \\ &= \pi \\ \theta &= \frac{\pi}{n+2} \end{aligned}$$

In general,

$$\theta_k = \frac{k\pi}{n+2}, \quad k = 1, 2, \dots, n. \quad (6)$$

Substituting equation (6) in (5),

$$\begin{aligned} \lambda_k &= 2 \cos 2\theta_k, \\ \lambda_k &= 2 \cos \left(\frac{2k\pi}{n+2} \right). \end{aligned} \quad (7)$$

Hence the equation (7) is called the spectrum of $A^{(2)}(P_n)$.

In the second order adjacency path graph P_n , each vertex v_n is adjacent in $A^{(2)}(P_n)$ only to the vertices v_{n-2} and v_{n+2} . Hence, vertices with odd indices are connected only to vertices with odd indices, vertices with even indices are connected only to vertices with even indices. Because second order adjacency in P_n becomes standard adjacency in these induced subgraphs,

$$(v_1, v_3, v_5, \dots; v_2, v_4, v_6, \dots),$$

For n is even, ($n = 2m$) the eigenvalues of the matrix is,
(8)

put $n = 2m$

$$\begin{aligned}\lambda_k &= 2 \cos \left(\frac{2k\pi}{2m+2} \right) \\ &= 2 \cos \left(\frac{2k\pi}{2(m+1)} \right) \\ &= 2 \cos \left(\frac{k\pi}{m+1} \right).\end{aligned}$$

For n is odd, ($n = 2m + 1$) then P_n decomposes into P_{m+1} and P_m give eigenvalues. That is,

$$A^{(2)}(P_n) = A(P_{m+1}) \cup A(P_m).$$

We know that, the adjacency eigenvalues of P_n are

$$\lambda_k = 2 \cos \left(\frac{k\pi}{n+1} \right), \quad k = 1, 2, \dots, n.$$

Put $n = m + 1$ for P_{m+1} , we get,

$$\lambda_k = 2 \cos \left(\frac{k\pi}{m+2} \right), \quad k = 1, 2, \dots, m+1.$$

Put $n = m$ for P_m , we get,

$$\lambda_k = 2 \cos \left(\frac{k\pi}{m+1} \right), \quad k = 1, 2, \dots, m.$$

Hence proved.

Example 2. Find the eigenvalues of its second-order adjacency matrix and energy for P_6 . Consider the path graph P_6 with vertex set $V(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

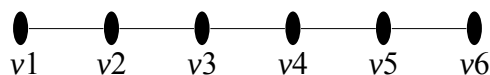


Figure 2: Path graph P_6

The second order adjacency matrix P_6 is defined by

$$A^{(2)}(P_6)_{ij} = \begin{cases} 1, & \text{if } d(v_i, v_j) = 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$A^{(2)}(P_6) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If $n = 6(2m)$ is even; where $2m = 6$. Therefore $m = 3$ is an integer. Then P_6 is into two graphs P_3 and P_3 . The eigenvalues are,

$$\lambda_k^{(2)} = 2 \cos \left(\frac{k\pi}{m+1} \right), \quad k = 1, 2, \dots, m.$$

$$\lambda_1^{(2)} = 2 \cos \left(\frac{\pi}{4} \right) = 1.414, \quad k = 1, 2, 3.$$

$$\lambda_2^{(2)} = 2 \cos \left(\frac{2\pi}{4} \right) = 0.$$

$$\lambda_3^{(2)} = 2 \cos \left(\frac{3\pi}{4} \right) = -1.414$$

Each with multiplicity 2. Therefore the second order adjacency energy is defined as,

$$\begin{aligned} E^{(2)}(P_n) &= \sum_{k=1}^n |\lambda_k| \\ E^{(2)}(P_6) &= \sum_{k=1}^3 |\lambda_k| \\ &= |2(1.414)| + |2(0)| + |2(-1.414)| \\ &= 5.656. \end{aligned}$$

The standard energy value is 6.988. So, $E^{(2)}(P_n) \leq E(P_n)$.

Example 3. Find the eigenvalues of second-order adjacency matrix and its energy for P_7 . Consider the path graph P_7 with vertex set $V(P_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

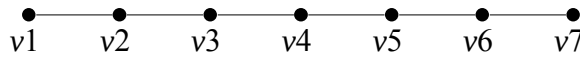


Figure 3: Path graph P_7

The second order adjacency matrix $A^{(2)}(P_7)$ is defined by

$$A^{(2)}(P_7)_{ij} = \begin{cases} 1, & \text{if } d(v_i, v_j) = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix is given by $A^{(2)}(P_7) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

If $n = 7(2m+1)$ is odd; where $2m+1 = 7$. Therefore $m = 3$. Then $A^{(2)}(P_7)$ is into two graphs P_4 and P_3 . The eigenvalues are, for $P_{m+1} = P_4$

$$\begin{aligned} \lambda_k^{(2)} &= 2 \cos \left(\frac{k\pi}{m+2} \right), \quad k = 1, 2, \dots, m+1, \\ \lambda_k^{(2)} &= 2 \cos \left(\frac{k\pi}{5} \right), \quad k = 1, 2, 3, 4. \\ \lambda_1^{(2)} &= 2 \cos \left(\frac{\pi}{5} \right) = 1.618; \quad \lambda_2^{(2)} = 2 \cos \left(\frac{2\pi}{5} \right) = 0.618. \\ \lambda_3^{(2)} &= 2 \cos \left(\frac{3\pi}{5} \right) = -0.618; \quad \lambda_4^{(2)} = 2 \cos \left(\frac{4\pi}{5} \right) = -1.618. \end{aligned}$$

For $P_m = P_3$,

$$\begin{aligned} \lambda_k^{(2)} &= 2 \cos \left(\frac{k\pi}{m+1} \right), \quad \text{for } k = 1, 2, \dots, m, \\ \lambda_k^{(2)} &= 2 \cos \left(\frac{k\pi}{4} \right), \quad \text{for } k = 1, 2, 3. \\ \lambda_1^{(2)} &= 2 \cos \left(\frac{\pi}{4} \right) = 1.414; \quad \lambda_2^{(2)} = 2 \cos \left(\frac{2\pi}{4} \right) = 0. \\ \lambda_3^{(2)} &= 2 \cos \left(\frac{3\pi}{4} \right) = -1.414. \end{aligned}$$

Therefore the eigenvalues are,

$$\lambda^{(2)}_k = \{1.618, 1.414, 0.618, 0, -1.618, -1.414, -0.618\}.$$

$$E^{(2)}(P_n) = \sum_{k=1}^n |\lambda_k|, \quad n \geq 3.$$

Therefore the second order adjacency energy is defined as,

$$E^{(2)}(P_7) = \sum_{k=1}^7 |\lambda_k| = 7.300$$

The standard energy value is 8.052. Therefore, $E^{(2)}(P_n) \leq E(P_n)$.

Theorem 4.2. Let C_n be the cycle graph with $V(G)$ be the vertex set and $E(G)$ be the edge set of G on $n \geq 4$ vertices. Then the second order adjacency energy of C_n is,

$$E^{(2)}(C_n) = \sum_{k=0}^{n-1} |\lambda_k^{(2)}(C_n)|$$

The eigenvalues are,

$$\lambda_k^{(2)} = 2 \cos \left(\frac{4k\pi}{n} \right), \quad k = 0, 1, \dots, n-1$$

Proof. Consider a cycle graph C_n with vertex set $\{v_1, v_2, \dots, v_n\}$, where vertices are taken modulo n . In C_n , each vertex v_i has exactly two vertices at distance two, namely v_{i-2} and v_{i+2} . Thus, the second order adjacency matrix $A^{(2)}(C_n) = [a_{ij}]$ is defined by

Hence the matrix is,

$$A^{(2)}(C_n) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

Therefore, $A^{(2)}(C_n)$ is a symmetric circulant matrix of width two. Let,

$$A^{(2)}(C_n)x_n = \lambda x_n \quad (1)$$

In C_n , the vertices at distance two from a vertex v_n are v_{n-2} and v_{n+2} whenever they exist. Component wise for each interior vertices,

$$x_{n-2} + x_{n+2} = \lambda x_n, n \geq 4 \quad (2)$$

This is a second order linear recurrence relation with step size 2. Assume,

$$xn = r^n \quad (3)$$

Substitute equation (3) in (2), we get

$$\begin{aligned} r^{n-2} + r^{n+2} &= \lambda r^n \\ \frac{1}{r^2} + r^2 &= \lambda \end{aligned} \quad (4)$$

Let us consider

$$\begin{aligned} r &= e^{i\theta} \\ r^2 &= e^{2i\theta} \\ \frac{1}{r^2} &= r^{-2} = e^{-2i\theta} \end{aligned} \quad (5)$$

Substitute equation (5) in (4),

$$\begin{aligned} e^{2i\theta} + e^{-2i\theta} &= \lambda \\ \cos 2\theta + i \sin 2\theta + \cos(-2\theta) + i \sin(-2\theta) &= \lambda \\ \lambda &= 2 \cos 2\theta \\ \lambda_k &= 2 \cos 2\theta_k \end{aligned} \quad (6)$$

Since C_n is a cycle, eigenvectors must satisfy the boundary conditions,

$$xn+1 = x1, \quad xn = x0 \quad (7)$$

Substitute (3) in (7),

$$\begin{aligned} r^{n+1} &= r \\ r^n &= 1 \Rightarrow e^{in\theta} = 1 \\ \cos n\theta &= 1 \Rightarrow n\theta = 2\pi \\ \theta &= \frac{2\pi}{n} \\ \theta_k &= \frac{2k\pi}{n} \end{aligned} \quad (8)$$

Substitute (8) in (6), we get

$$\lambda_k = 2 \cos \left(\frac{4k\pi}{n} \right), \quad k = 0, 1, \dots, n-1 \quad (9)$$

Then the second order adjacency energy of C_n is,

$$E^{(2)}(C_n) = \sum_{k=0}^{n-1} \left| \lambda_k^{(2)}(C_n) \right|.$$

This completes the proof.

Example 4. Find the eigenvalues of $A^{(2)}(C_8)$ and its energy.
The second order adjacency matrix of C_8 is

$$A^{(2)}(C_8) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of C_8 are given by Here $n = 8$,

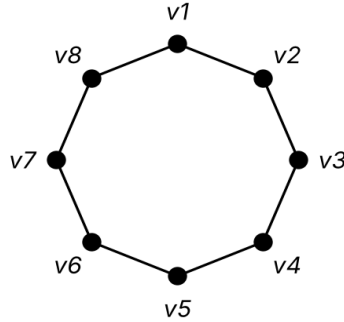


Figure 4: Cycle graph C_8

$$\lambda_k^{(2)} = 2 \cos \left(\frac{4k\pi}{n} \right), \quad k = 0, 1, \dots, n-1.$$

$$\lambda_k^{(2)} = 2 \cos \left(\frac{4k\pi}{8} \right), \quad k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$\lambda_0^{(2)} = 2 \cos(0) = 2; \quad \lambda_1^{(2)} = 2 \cos \left(\frac{4\pi}{8} \right) = 0.$$

$$\lambda_2^{(2)} = 2 \cos \left(\frac{8\pi}{8} \right) = -2; \quad \lambda_3^{(2)} = 2 \cos \left(\frac{12\pi}{8} \right) = 0.$$

$$\lambda_4^{(2)} = 2 \cos \left(\frac{16\pi}{8} \right) = 2; \quad \lambda_5^{(2)} = 2 \cos \left(\frac{20\pi}{8} \right) = 0.$$

$$\lambda_6^{(2)} = 2 \cos \left(\frac{24\pi}{8} \right) = -2; \quad \lambda_7^{(2)} = 2 \cos \left(\frac{28\pi}{8} \right) = 0.$$

Then the eigenvalues are

$$\lambda_k^{(2)} = \{2, 2, 0, 0, 0, 0, -2, -2\}.$$

Then the second order adjacency energy is,

$$E^{(2)}(C_n) = \sum_{k=0}^{n-1} |\lambda_k^{(2)}|.$$

$$E^{(2)}(C_8) = 2(2) + 2(2) = 8.$$

The standard energy value is 8. Therefore $E^{(2)}(C_n) = E(C_n)$.

Example 5. Compute the eigenvalues of C_5 and its energy.

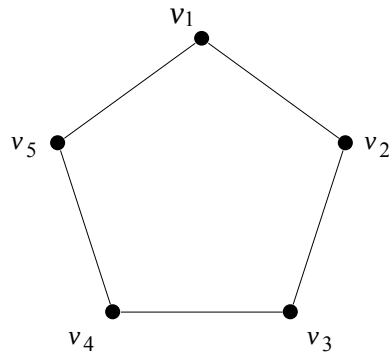


Figure 5: Cycle graph C_5

The matrix $A^{(C_5)}$ is

$$A^{(2)}(C_5) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The eigenvalues of C_5 are given by Here $n = 5$. Therefore the eigenvalues are,

$$\lambda_k^{(2)} = 2 \cos \left(\frac{4k\pi}{n} \right), \quad k = 0, 1, \dots, n-1$$

$$\lambda_k^{(2)} = 2 \cos \left(\frac{4k\pi}{5} \right), \quad k = 0, 1, 2, 3, 4.$$

$$\lambda_0^{(2)} = 2 \cos(0) = 2,$$

$$\lambda_1^{(2)} = 2 \cos \left(\frac{4\pi}{5} \right) = -1.618,$$

$$\lambda_2^{(2)} = 2 \cos \left(\frac{8\pi}{5} \right) = 0.618,$$

$$\lambda_3^{(2)} = 2 \cos \left(\frac{12\pi}{5} \right) = 0.618,$$

$$\lambda_4^{(2)} = 2 \cos \left(\frac{16\pi}{5} \right) = -1.618.$$

The eigenvalues are,

$$\lambda_k^{(2)} = \{2, 1.618, 0.618, -1.618, -1.618\}$$

Then the second order adjacency energy is,

$$E^{(2)}(C_n) = \sum_{k=0}^{n-1} |\lambda_k^{(2)}|.$$

$$\begin{aligned} E^{(2)}(C_5) &= 2|0.618| + 2|1.618| + 2 \\ &= 6.472. \end{aligned}$$

The standard energy value is 6.472. Therefore, $E^{(2)}(C_n) = E(C_n)$.

Theorem 4.3. Let $S_n = \{K_{1,n-1}\}$ be the star graph on $n \geq 3$ vertices, then we have $E^{(2)}(S_n) = 2(n-2)$.

Proof. Consider S_n be a star graph with one central vertex and $n-1$ pendant vertices. Any two pendant vertices are at distance 2, while the central vertex is at distance 1 from all other vertices.

Hence, the second order adjacency matrix of S_n has a zero row and column corresponding to the central vertex and the sub matrix induced by the pendant vertices is the adjacency matrix of the complete graph K_{n-1} . Thus,

$$A^{(2)}(S_n) = \begin{pmatrix} 0 & 0 \\ 0 & A(K_{n-1}) \end{pmatrix}.$$

$$A^{(2)}(S_n) = \begin{pmatrix} 0 & 0^T \\ 0 & J_{n-1} - I_{n-1} \end{pmatrix},$$

$$J_{n-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad I_{n-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The eigenvalues of $A(K_{n-1})$ are $n-2$ and -1 with multiplicity $n-2$. Including the zero eigenvalue from the central vertex, we obtain

$$\text{Spec}(A^{(2)}(S_n)) = \{n-2, -1^{(n-2)}, 0\}.$$

Therefore, the second order adjacency energy of S_n is

$$E^{(2)}(S_n) = |n-2| + (n-2)|-1| = 2(n-2).$$

Example 6. To illustrate the example of $S_7 = \{K_{1,6}\}$ to compute the eigenvalues of $A^{(2)}(S_7)$ and its energy.

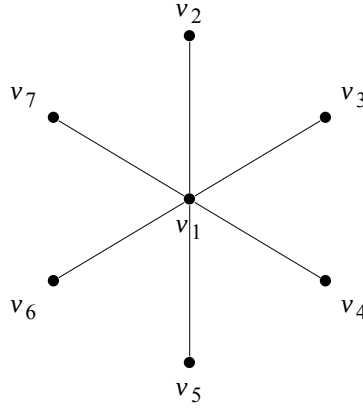
The Second order adjacency matrix of S_7

$$A^{(2)}(S_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Here the characteristic polynomial is,

$$[\lambda^2 - (n-2)\lambda](\lambda+1)^{(n-2)} = 0.$$

$$\text{Here, } n = 7, [\lambda^2 - 5\lambda](\lambda+1)^5 = 0.$$

Figure 6: Star graph S_7

The spectrum (set of all eigenvalues) is defined as follows:

$$A^{(2)}(\text{Spec}(S_7)) = \begin{pmatrix} 5 & -1 \\ 1 & 5 \end{pmatrix}$$

Hence the second order adjacency energy is, $E^{(2)}(S_n) = 2(n - 2)$

$$E^{(2)}(S_7) = 10.$$

The standard adjacency energy for S_7 is,

$$E(S_7) = 4.90.$$

Hence, $E(S_7) \leq E^{(2)}(S_7)$.

Theorem 4.4. Let K_n be the complete graph on $n \geq 2$ vertices. Then $A^{(2)}(K_n) = 0$. Consequently, $E^{(2)}(K_n) = 0$.

Proof. In the complete graph K_n , every pair of distinct vertices v_i and v_j with $d(v_i, v_j) = 1$.

Since there are no vertex pairs at distance two, by the definition of second order adjacency matrix. Then we have

$$a_{ij}^{(2)} = 0 \quad \text{for all } i, j.$$

Therefore,

$$A(2)(K_n) = 0_n,$$

where 0_n is the n^{th} zero matrix.

Hence, all eigenvalues of the zero matrix are 0, then the spectrum of the second-order adjacency matrix is,

$$\text{Spec}(A^{(2)}(K_n)) = 0.$$

As $\lambda_i = 0$ for all i , we obtain

$$E^{(2)}(K_n) = 0.$$

Hence, second order adjacency energy of the complete graph is zero.

Remark:

- For the complete graph K_n , the standard adjacency energy is

$$E(K_n) = 2(n - 1),$$

whereas the second-order adjacency energy is zero.

- This highlights a fundamental structural difference between the adjacency and second-order adjacency matrices.

Observation Table:

From the following table, it is observed that the second-order adjacency energy behaves differently based on the structure of the graph.

Graph	Relation between $E^{(2)}(G)$ and $E(G)$
P_n	$E^{(2)}(P_n) \leq E(P_n)$
C_n	$E^{(2)}(C_n) = E(C_n)$
S_n	$E^{(2)}(S_n) \geq E(S_n)$
K_n	$E^{(2)}(K_n) = 0$

Table 1: Comparison of Standard and Second Order Adjacency Energies

5. Application

- The second-order adjacency matrix captures interactions between vertices at distance two. Unlike the ordinary adjacency matrix, which measures immediate neighborhood connectivity reflects indirect structural influence.
- Large positive eigenvalues indicate dense second-neighbor connectivity and zero eigenvalues reflect absence of distance-two relations. Thus the second-order energy quantifies the strength of indirect interactions in a graph.
- Second-order adjacency models two-step diffusion. This energy reflects total diffusion potential.
- It is used to find graphs with maximum or minimum energy. It is easy to compare the families of graphs.

6. Conclusion

In this paper, the concept of the second order adjacency matrix for standard graphs and investigated its associated spectral properties. Explicit constructions of the second order adjacency matrix were presented for path (P_n), cycle (C_n), star (S_n) graphs and complete graphs. Using these matrices, the corresponding eigenvalues and energy values were computed and analysed.

Future work may extend this study to other graph families and explore bounds and extremal properties of second order adjacency energy.

Applications to network analysis and chemical graph theory may also be investigated.

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