

Fixed Point Theorems For New Contractions On B-Metric Spaces

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Abstract

The aim of this paper is to introduce new contractions which admit to amend and generalize some results earned by Zheng et al. (Zheng?). We also present an example corresponding to this new contraction of our main result. Further, we discuss an application to the solvability of nonlinear equations.

Keywords: fixed point; b-metric space; Gregus type $\theta - \phi$ contraction.

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1. Introduction

The Banach contraction principle (**Banach**) is a fundamental result in fixed point theory. Due to its importance, various mathematician studied many interesting extensions and generalizations, [**Browder**; **Kari**; **Mlaiki**; **Roshan**]. Jleli and Samet [**Jleli**] analyzed a generalization of Banach fixed point theorem on generalized metric spaces in new type of contraction mappings called (θ) -contraction mapping (or J-S contraction) and proved a fixed point result in generalized metric spaces. This direction has been studied and various fixed point theorems have been developed [**Kari1**; **Kari3**]. Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b-metric spaces were introduced by Bakhtin [**Bakhtin**] and Czerwik [**Czerwik**], in such a way that triangle inequality is replaced by the b-triangle inequality. Any metric space is a b-metric space but in general, b-metric space might not be a metric space. Various fixed point results were established on such spaces. For more information on b-metric spaces and b-metric like-spaces, the readers can refer to [**Demir**; **Jleli**; **Nashine**; **Reich**]. Very recently, Zheng et al. [**Zheng**] introduced a new concept of $(\theta - \phi)$ -contraction

and established some fixed point results for such mappings in complete metric space and generalized the results of Browder and Kannan [**Browder**].

2 Preliminaries

We, provide several definitions and results that are helpful to understand the main section.

Definition 1. (Czerwik) : Let (X, d) be a b -metric space with coefficient $s \geq 1$ be a given real number. A function $d: X \times X \rightarrow \mathbb{R}^+$ is a b -metric space if and only if for all $x, y, z \in X$, the following conditions are satisfied :

(I) $d(x, y) = 0$ if and only if $x = y$;

(II) $d(x, y) = d(y, x)$;

(III) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b -metric space.

Lemma 2. (Aghajani) Let (X, d) be a b -metric space.

(a) suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have $\frac{1}{s^2} d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2 d(x, y)$. (b) In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have $\frac{1}{s} d(x, z) \leq \lim_{n \rightarrow \infty} \inf d(x_n, z) \leq \lim_{n \rightarrow \infty} \sup d(x_n, z) \leq s d(x, z)$, for all $x \in X$.

Lemma 3. (Czerwik) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequence $\{m(k)\}$ and $\{n(k)\}$ of a positive integers such that $\epsilon \leq$

$$\liminf_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) \leq s\epsilon, \epsilon \leq$$

$$\liminf_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)+1}) \leq \limsup_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)+1}) \leq s\epsilon, \epsilon \leq$$

$$\liminf_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)+1}) \leq s\epsilon, \frac{\epsilon}{s} \leq$$

$$\liminf_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^2\epsilon.$$

The following definition was given by Jleli et al. (**Jleli**).

Definition 4. (Jleli) Let Θ be a family of all functions $\theta:]0, +\infty[\rightarrow]1, +\infty[$ such that (θ_1) θ is a increasing,

(θ_2) for each sequence $\{x_n\} \subset]0, +\infty[$; $\lim_{n \rightarrow 0} x_n = 0$ if and only if $\lim_{n \rightarrow \infty} \theta(x_n) = 1$;

(θ_3) θ is a continuous.

Zheng et al. (**Zheng**), presented the concept of $(\theta - \phi)$ - contraction on metric spaces and proved the following result.

Definition 5. (Zheng) Let Φ be a family of all functions $\Phi: [1, +\infty[\rightarrow [1, +\infty[$, such that

(ϕ_1) ϕ is non-decreasing,

(ϕ_2) for each $t \in]1, +\infty[$, $\lim_{n \rightarrow \infty} \phi^n(t) = 1$;

(ϕ_3) ϕ is a continuous.

Lemma 6. (Zheng) If $\phi \in \Phi$, then $\phi(1) = 1$, and $\phi(t) < t \forall t \in]1, \infty[$. In this paper, we introduce new notions of Gregus type $(\theta) -$ and Gregus type $(\theta -$

ϕ) –contraction (or R-N contraction) and establish some results of Zheng et al. (**Zheng**), Kannan (**Kannan**) and Reich (**Reich**) on b- metric space. An example is constructed to illustrate our result. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations.

3 Main Result

In this part, we discuss new results in the context of Gregus type θ and Gregus type (θ, ϕ) -contractions.

Definition 7. (*Jleli*) Let (X, d) be a metric space with parameter $s > 1$ space and $T: X \rightarrow X$ be a mapping.

T is said to be Gregus type θ -contraction if there exist $\theta \in \Theta$ and $r \in]0,1[$ such that $d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r$, where $M(x, y) = a d(x, y) + (1 - a) \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2} \right\}$

Definition 8. Let (X, d) be a metric space with parameter $s > 1$ space and $T: X \rightarrow X$ be a mapping.

T is said to be Gregus type $(\theta - \phi)$ -contraction if there exist $\theta \in \Theta$ and $r \in]0,1[$ such that $d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \phi[\theta(M(x, y))]$, where $M(x, y) = a d(x, y) + (1 - a) \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2} \right\}$.

Theorem 9. Let (X, d) be a complete b-metric space and $T: X \rightarrow X$ be a Gregus type θ -contraction and $0 \leq a < 1$ i.e. there exist $\theta \in \Theta$ and $r \in]0,1[$ such that for any $x, y \in X$, we have $d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$, $\forall n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ then the proof is completed. We can suppose that $d(x_n, x_{n+1}) > 0$, $\forall n \in \mathbb{N}$. Letting $x = x_{n-1}$ and $y = x_n$ in [eq1], we have

$$\theta[d(x_n, x_{n+1})] \leq \theta[s^3 d(x_n, x_{n+1})] \leq [\theta(M(x_{n-1}, x_n))]^r, \forall n \in \mathbb{N},$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= a d(x_{n-1}, x_n) + (1 - a) \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \right. \\ &\quad \left. \frac{d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n)}{2s^2} \right\} \\ &= a d(x_{n-1}, x_n) + (1 - a) \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s^2} \right\} \\ &= a d(x_{n-1}, x_n) + (1 - a) \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s^2} \right\}. \end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2s^2}d(x_{n-1}, x_{n+1}) &\leq \frac{1}{2s^2} \cdot s [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&= \frac{1}{2s} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\
&\leq \frac{1}{2s} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\
&\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},
\end{aligned}$$

we obtain

$$M(x_{n-1}, x_n) = ad(x_{n-1}, x_n) + (1-a)\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ by [eq2], we have

$$\begin{aligned}
\theta(d(x_n, x_{n+1})) &\leq (\theta(d(x_n, x_{n+1})))^r \\
&< (\theta(d(x_n, x_{n+1}))),
\end{aligned}$$

which is a contradiction. Hence $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^r.$$

Repeating this step, we conclude that

$$\begin{aligned}
\theta(d((x_n, x_{n+1}))) &\leq (\theta(d(x_{n-1}, x_n)))^r \\
&\leq (\theta(d(x_{n-2}, x_{n-1})))^{r^2} \\
&\vdots \\
&\leq (\theta(d(x_0, x_1)))^{r^n}.
\end{aligned}$$

From [eq3] and using (θ_1) , we get $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. Therefore, $d(x_n, x_{n+1})_{n \in \mathbb{N}}$ is monotone strictly decreasing sequence of non-negative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \alpha.$$

Now we claim that $\alpha = 0$. Arguing by contradiction, we assume that $\alpha > 0$. Since $d(x_n, x_{n+1})_{n \in \mathbb{N}}$ is a non-negative decreasing sequence, we have $d(x_n, x_{n+1}) \geq \alpha, \forall n \in \mathbb{N}$. By the property of (θ_1) , we get

$$1 < \theta(\alpha) \leq \theta(d(x_0, x_1))^{r^n}.$$

Letting $n \rightarrow \infty$ in [eq4], we obtain $1 < \theta(\alpha) \leq 1$. This is a contradiction. Therefore,

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Next, we shall prove that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. By Lemma 2.3, there is an $\epsilon > 0$ such that for an integer k there exists two sequence $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that

- (i) $\epsilon \leq \lim_{k \rightarrow \infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon;$
- (ii) $\frac{\epsilon}{s} \leq \lim_{k \rightarrow \infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s^2\epsilon;$
- (iii) $\frac{\epsilon}{s} \leq \lim_{k \rightarrow \infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s^2\epsilon;$

$$(iv) \quad \frac{\epsilon}{s^2} \leq \lim_{k \rightarrow \infty} \inf d(x_{m(k)+1}, x_{n(k)+1}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3 \epsilon.$$

From [eq1] and by setting $x = x_{m(k)}$ and $y = x_{n(k)}$ we have

$$M(x_{m(k)}, x_{n(k)}) = \max \left\{ d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \frac{1}{2s^2} (d((x_{n(k)}, x_{m(k)+1}))) \right\}.$$

Taking the limit as $k \rightarrow \infty$ and using [eq5] and Lemma 2.3, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}) &= \lim_{k \rightarrow \infty} \max \{ d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\quad \frac{1}{2s^2} (d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)+1})) \} \\ &\leq \max \left\{ s\epsilon, 0, 0, \frac{1}{2s^2} (s^2\epsilon + s^2\epsilon) \right\} \\ &= a d(x_{m(k)}, x_{n(k)}) + (1-a) \max \left\{ s\epsilon, 0, 0, \frac{2s^2\epsilon}{2s^2} \right\} \\ &= a d(x_{m(k)}, x_{n(k)}) + (1-a)s\epsilon \\ &= a(s\epsilon) + (1-a)s\epsilon \\ &= s\epsilon. \end{aligned}$$

So, we have

$$\lim_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}) \leq s\epsilon.$$

Now, letting $x = x_{m(k)}$ and $y = x_{n(k)}$ [eq1], we obtain

$$\theta \left[s^3 (x_{m(k)+1}, x_{n(k)+1}) \right] \leq \left[\theta (M(x_{m(k)}, x_{n(k)})) \right]^r.$$

Letting $k \rightarrow \infty$ the above inequality, applying the continuity of θ and using in [eq6], we obtain

$$\begin{aligned} \theta \left(\frac{\epsilon}{s^2} \cdot s^3 \right) &= \theta(s\epsilon) \\ &\leq \theta \left(s^3 \lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}) \right) \\ &\leq \left[\theta \left(\lim_{k \rightarrow \infty} M(x_{m(k)+1}, x_{n(k)+1}) \right) \right]^r. \end{aligned}$$

Therefore,

$$\theta(s\epsilon) \leq [\theta(s\epsilon)]^r < \theta(s\epsilon).$$

Since θ is increasing, we get $s\epsilon < s\epsilon$ which is contradiction. Thus $\lim_{n,m \rightarrow \infty} d(x_m, x_n) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X . By completeness of (X, d) there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} d(x_n, z) = 0.$$

Now, we show that $d(Tz, z) = 0$, by contradiction. Assume that $d(Tz, z) > 0$. Since $x_n \rightarrow z$ as $n \rightarrow \infty$, from Lemma 2.2, we conclude that

$$\frac{1}{2s^2} d(z, Tz) \leq \limsup_{n \rightarrow \infty} d(Tx_n, Tz) \leq s^2 d(z, Tz).$$

Now letting $x = x_n$ and $y = z$ in [eq1], we have

$$\theta(s^3 d(Tx_n, Tz)) \leq [\theta(M(x_n, z))]^r, \forall n \in \mathbb{N},$$

where

$$M(x_n, z) = a d(x_n, z) + (1 - a) \max \left\{ d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2} (d(z, Tx_n) + d(x_n, Tz)) \right\}.$$

Taking the limit as $n \rightarrow \infty$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} M(x_n, z) &= \limsup_{n \rightarrow \infty} \max \left\{ d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2} (d(z, Tx_n) + d(x_n, Tz)) \right\} \\ &= d(z, Tz). \end{aligned}$$

Therefore

$$\begin{aligned} &\theta(s^3 d(Tx_n, Tz)) \\ &\leq [\theta(\max \left\{ d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2} (d(z, Tx_n) + d(x_n, Tz)) \right\})]^r. \end{aligned}$$

Taking $n \rightarrow \infty$ in [eq7], [eq5] and (θ_3) , we obtain

$$\begin{aligned} \theta[s^3 \cdot \frac{1}{s} d(z, Tz)] &= \theta[s d(z, Tz)] \\ &\leq \theta[s^3 \lim_{n \rightarrow \infty} d(Tx_n, Tz)] \\ &\leq [\theta(d(z, Tz))]^r \\ &< \theta(d(z, Tz)). \end{aligned}$$

By (θ_1) , we get $sd(z, Tz) < d(z, Tz)$. This implies that $d(z, Tz)(s - 1) < 0 \Rightarrow s < 1$, which is a contradiction. Hence $Tz = z$. Now suppose that $z, u \in X$ are two fixed point of T Such that $u \neq z$. Then we have $d(z, u) = d(Tz, Tu) > 0$. Letting $x = z$ and $y = u$ in [eq1] we have

$$\begin{aligned} \theta(d(z, u)) &= \theta(d(Tu, Tz)) \\ &\leq \theta(s^3 d(Tu, Tz)) \\ &\leq [\theta(M(z, u))]^r, \end{aligned}$$

where

$$\begin{aligned} M(z, u) &= a d(z, u) + (1 - a) \max \left\{ d(z, u), d(z, Tz), d(u, Tu), \frac{1}{2s^2} (d(u, Tu) + d(z, Tu)) \right\} \\ &= ad(z, u) + (1 - a)d(z, u). \end{aligned}$$

Therefore, we have

$$\theta(d(z, u)) \leq [\theta(d(z, u))]^r < \theta(d(z, u)),$$

which is a contradiction. Therefore $u = z$. $\square$

For $a = 1$ in special case, we get the following :

Corollary 3.4 Let (X, d) be a complete b- metric space and $T: X \rightarrow X$ be a given mapping. Suppose that there exists $\theta \in \Theta$ and $k \in]0, 1[$ such that for any $x, y \in X$. We have

$$\begin{aligned} & d(Tx, Ty) > 0 \implies \theta[s^3 d(Tx, Ty)] \\ & \leq \theta[d(x, y)]^k \end{aligned}$$

Then T has a unique fixed point.

Example: Let $X = [1, +\infty[$. Define $d: X \times X \rightarrow [0, \infty[$ by $d(x, y) = |x - y|^2$. Then (X, d) is a b-metric space with coefficient $s = 2$. Define a mapping $T: X \rightarrow X$ by

$T(x) = x^{\frac{1}{8}}$. Evidently, $T(x) \in X$. Let $\theta(t) = e^{\sqrt{t}}$, $r = \frac{1}{\sqrt{4}}$. It is obvious that $\theta \in \Theta$ and $r \in]0, 1[$. Consider the following possibilities:

1. $x, y \in [1, \infty[, y < x$. Then $T(x) = x^{\frac{1}{8}}, T(y) = y^{\frac{1}{8}}, d(Tx, Ty) = (x^{\frac{1}{8}} - y^{\frac{1}{8}})^2$.

On the other hand $\theta[s^3 d(Tx, Ty)] = e^{\sqrt{16}(x^{\frac{1}{8}} - y^{\frac{1}{8}})}$ and

$$\begin{aligned} M(x, y) &= ad(x, y) + (1 - a) \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s^2} (d(y, Tx) + d(x, Ty)) \right\} \\ &= ad(x, y) + (1 - a)d(x, y) \\ &\geq (x - y)^4 \\ &= [(x^{\frac{1}{8}} - y^{\frac{1}{8}}) \cdot (x^{\frac{1}{8}} + y^{\frac{1}{8}}) \cdot (x^{\frac{1}{4}} + y^{\frac{1}{4}})]^2. \\ &\geq [8(x^{\frac{1}{8}} - y^{\frac{1}{8}})]^2. \end{aligned}$$

Hence

$$\begin{aligned} [\theta(d(x, y))]^{\frac{1}{2}} &= \left[e^{[(x^{\frac{1}{8}} - y^{\frac{1}{8}})(x^{\frac{1}{8}} + y^{\frac{1}{8}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})]} \right]^{\frac{1}{2}} \\ &\geq \left[4(x^{\frac{1}{8}} - y^{\frac{1}{8}}) \right]^{\frac{1}{2}} \\ &= \left[4(x^{\frac{1}{8}} - y^{\frac{1}{8}}) \right]^4. \end{aligned}$$

This implies that

$$\begin{aligned}\theta(s^3 d(Tx, Ty)) &\leq \phi[\theta(d(x, Tx))]^{\frac{1}{\sqrt{2}}} \\ &\leq [\theta(\max\{d(x, y), d(x, Tx), d(y, Ty)\}, d(y, Tx))]^{\frac{1}{\sqrt{2}}}.\end{aligned}$$

2. $x < y$ with $x, y \in [1, \infty[$. By a similar method, we conclude that

$$\begin{aligned}\theta(s^3 d(Tx, Ty)) &\leq \phi[\theta(d(x, Tx))]^{\frac{1}{\sqrt{2}}} \\ &\leq [\theta(\max\{d(x, y), d(x, Tx), d(y, Ty)\}, d(y, Tx))]^{\frac{1}{\sqrt{2}}}.\end{aligned}$$

Hence, the condition [eq1] is satisfied. Therefore, T has a unique fixed point $z = 1$.

Theorem 10. Let (X, d) be a complete b-metric space and $T: X \times X$ be a mapping. Suppose that there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$, $d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \phi[\theta(M(x, y))]$, where $M(x, y) = a d(x, y) + (1 - a) \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2}\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and we define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \forall n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then proof is completed. We can suppose that

$$d(x_n, x_{n+1}) > 0 \forall n \in \mathbb{N}.$$

Letting $x = x_{n-1}$ and $y = x_n$ in [eq8], we have

$$\theta[d(x_n, x_{n+1})] \leq \theta[s^3 d(x_n, x_{n+1})] \leq \phi[\theta(M(x_{n-1}, x_n))], \forall n \in \mathbb{N}.$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= a d(x_{n-1}, x_n) + (1 - a) \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2s^2}\} \\ &= a d(x_{n-1}, x_n) + (1 - a) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{d(x_{n-1}, x_{n+1})}{2s^2}\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s^2} d(x_{n-1}, x_{n+1}) &\leq \frac{1}{2s^2} [sd(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= \frac{1}{2s} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &\leq \frac{1}{2s} (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},\end{aligned}$$

we obtain

$$M(x_{n-1}, x_n) = a d(x_{n-1}, x_n) + (1 - a) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ then by [eq9], we have

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \phi(\theta(d(x_n, x_{n+1}))) \\ &< (\theta(d(x_n, x_{n+1}))),\end{aligned}$$

which is a contradiction. Hence $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus $\theta(d(x_n, x_{n+1})) \leq \phi(\theta(d(x_{n-1}, x_n)))$ Repeating this step, we conclude that

$$\begin{aligned}\theta(d((x_n, x_{n+1}))) &\leq \phi(\theta(d(x_{n-1}, x_n))) \\ &\leq \phi^2(\theta(d(x_{n-2}, x_{n-1}))) : \\ &\leq \phi^n(\theta(d(x_0, x_1))).\end{aligned}$$

From [eq3] and using Lemma 2.3 and (θ_1) , we get $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. Therefore, $d(x_n, x_{n+1})_{n \in \mathbb{N}}$ is monotone strictly decreasing sequence of non-negative real numbers. Consequently, there exists $\alpha \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \alpha$. Now, we claim that $\alpha = 0$. Assume that $\alpha > 0$. Since $d(x_n, x_{n+1})_{n \in \mathbb{N}}$ is a non-negative decreasing sequence, we have $d(x_n, x_{n+1}) \geq \alpha, \forall n \in \mathbb{N}$.

This implies that

$$\begin{aligned}1 < \theta(\alpha) &\leq \theta(d(x_{n+1}, x_n)) \\ &\leq \phi[\theta(d(x_n, x_{n-1}))] \\ &\vdots \\ &\leq \phi^n[\theta(d(x_0, x_1))]\end{aligned}$$

Letting $n \rightarrow \infty$ and using the properties of ϕ and θ , we get

$$1 < \theta(\alpha) \leq \phi^n[\theta(d(x_0, x_1))] = 1,$$

which is a contradictions. Thus $\alpha = 0$ and so, we have $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Next we shall prove that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, i.e. $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. By Lemma 2.3 there is an $\epsilon > 0$ such that for an integer k there exists two sequence $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that

- (i) $\epsilon \leq \lim_{k \rightarrow \infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon;$
- (ii) $\frac{\epsilon}{s} \leq \lim_{k \rightarrow \infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s^2\epsilon;$
- (iii) $\frac{\epsilon}{s} \leq \lim_{k \rightarrow \infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s^2\epsilon;$
- (iv) $\frac{\epsilon}{s^2} \leq \lim_{k \rightarrow \infty} \inf d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq \lim_{k \rightarrow \infty} \sup d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq s^3\epsilon.$

From [eq8] and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\lim_{k \rightarrow \infty} M(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon.$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in [eq8], we obtain

$$\theta[s^3 d(x_{m_{(k)+1}}, x_{n_{(k)+1}})] \leq \phi[\theta(M(x_{m_{(k)}}, x_{n_{(k)}}))].$$

Letting $k \rightarrow \infty$ in the above inequality from the continuity of θ and ϕ and using [\[eq10\]](#), we obtain

$$\begin{aligned} \theta\left(\frac{\epsilon}{s^2} \cdot s^3\right) &= \theta(\epsilon s) \\ &\leq \theta\left(s^3 \lim_{k \rightarrow \infty} d(x_{m_{(k)+1}}, x_{n_{(k)+1}})\right) \\ &\leq \phi\left[\theta\left(\lim_{k \rightarrow \infty} M(x_{m_{(k)}}, x_{n_{(k)}})\right)\right]. \end{aligned}$$

By Lemma 2.2, we get $\theta(s\epsilon) \leq \phi[\theta(s\epsilon)] < \theta(s\epsilon)$. Since θ is increasing, we get $s\epsilon < s\epsilon$ which is a contradiction. Thus $\lim_{n,m \rightarrow \infty} d(x_m, x_n) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X . By completeness of (X, d) there exists $z \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, z) = 0$. Now we show that $d(Tz, z) = 0$. In contrary, let $d(Tz, z) > 0$. Since $x_n \rightarrow z$ as $n \rightarrow \infty$, from Lemma 2.2, we conclude that

$$\frac{1}{2s^2} d(z, Tz) \leq \limsup_{n \rightarrow \infty} d(Tx_n, Tz) \leq s^2 d(z, Tz).$$

Now letting $x = x_n$ and $y = z$ in [\[eq8\]](#), we have

$$\theta(s^3 d(Tx_n, Tz)) \leq [\theta(M(x_n, z))]^r, \forall n \in \mathbb{N},$$

Where

$$\begin{aligned} M(x_n, z) &= a d(x_n, z) + (1 - a) \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2} (d(z, Tx_n) \\ &\quad + d(x_n, Tz))\}. \end{aligned}$$

As in the proof of Theorem 2.3, we have $\lim_{n \rightarrow \infty} \sup M(x_n, z) = d(z, Tz)$. Therefore

$$\begin{aligned} &\theta(s^3 d(Tx_n, Tz)) \\ &\leq a d(x_n, z) + (1 - a) \left[\theta \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2} (d(z, Tx_n) \right. \\ &\quad \left. + d(x_n, Tz))\} \right]. \end{aligned}$$

Letting $n \rightarrow \infty$ in [\[eq11\]](#) and using the properties of ϕ and θ , we obtain

$$\begin{aligned} \theta\left[s^3 \frac{1}{s} d(z, Tz)\right] &= \theta[s d(z, Tz)] \\ &\leq \theta\left[s^3 \lim_{n \rightarrow \infty} d(Tx_n, Tz)\right] \\ &\leq \phi[\theta(d(z, Tz))] \\ &< \theta(d(z, Tz)). \end{aligned}$$

By (θ_1) , we get $sd(z, Tz) < d(z, Tz)$. This implies that $d(z, Tz)(s - 1) < 0 \Rightarrow s < 1$, which is a contradiction. Hence $Tz = z$.

Now suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$d(z, u) = d(Tz, Tu) > 0.$$

Letting $x = z$ and $y = u$ in [eq8], we have

$$\begin{aligned}\theta(d(z, u)) &= \theta(d(Tu, Tz)) \\ &\leq \theta(S^3(d(Tu, Tz))) \\ &\leq \phi[\theta(M(z, u))],\end{aligned}$$

where

$$\begin{aligned}M(z, u) &= ad(z, u) + (1 - a)\max\left\{d(z, u), d(z, Tz), d(u, Tu), \frac{1}{2s^2}(d(u, Tu) + d(z, Tu))\right\} \\ &= ad(z, u) + (1 - a)d(z, u) \\ &= d(z, u).\end{aligned}$$

Therefore, we have

$$\theta(d(z, u)) \leq \phi[\theta(d(z, u))] < \theta(d(z, u)).$$

which implies that $d(z, u) < d(z, u)$. This is a contradiction. Therefore $u = z$. $\square$

It follows from Theorem 3.4 that we coin the followed fixed point theorems for Gregus-Kannan $(\theta - \phi)$ type and Gregus-Reich $(\theta - \phi)$ - type. The results presented in the paper improve and extend the corresponding results due to Kannan - type contraction and Reich-type contraction on b-metric space.

Theorem 11. Let (X, d) be a complete b-metric space and $T: X \rightarrow X$ be a Gregus-Kannan $(\theta - \phi)$ type contraction. Then T has a unique fixed point.

Proof. Since T is a Gregus-Kannan $(\theta - \phi)$ type contraction, there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\begin{aligned}\theta[s^3d(Tx, Ty)] &\leq \phi\left[\theta\left(\frac{d(Tx, x) + d(Ty, y)}{2}\right)\right] \\ &\leq \phi[\theta(\max\{d(x, Tx), d(y, Ty)\})] \\ &\leq ad(x, y) + (1 - a)\phi\left[\theta\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{(d(y, Tx) + d(x, Ty))}{2s^2}\right\}\right)\right].\end{aligned}$$

Therefore, T is a Gregus $(\theta - \phi)$ contraction. As in the proof of Theorem 2.2, we conclude that T has a unique fixed point. $\square$

Theorem 12. Let (X, d) be a complete b-metric space and $T: X \rightarrow X$ be a Gregus-Reich $(\theta - \phi)$ type contraction. Then T has a unique fixed point.

Proof. Since T is Gregus-Reich $(\theta - \phi)$ type contraction, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\begin{aligned}
\theta[s^3 d(Tx, Ty)] &\leq \phi \left[\theta \left(\frac{d(x, y) + d(Tx, x) + d(Ty, y)}{3} \right) \right] \\
&\leq \phi \left[\theta \left(\max \{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s^2} (d(y, Tx) + d(x, Ty))\} \right) \right] \\
&\leq a d(x, y) + (1 - a) \phi \left[\theta \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{(d(y, Tx) + d(x, Ty))}{2s^2} \right\} \right) \right].
\end{aligned}$$

Therefore, T is a Gregus type $(\theta - \phi)$ -contraction. As in the proof of Theorem 2.3, we conclude that T has a unique fixed point. $\square$

4. Application to nonlinear integral equations

In this section, we endeavor to apply Theorems 3.4 and 3.6 to prove the existence and uniqueness of the integral equation of Fredholm type

$$x(t) = \lambda \int_a^b K(t, r, x(r)) ds,$$

where $a, b \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$ and $K: [a, b]^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function.

Theorem 13. Consider the nonlinear integral [\[eq12\]](#) and assume that the Kernel function K satisfies the condition $|K(t, r, x(r)) - K(t, r, y(r))| \leq \frac{1}{s^2} (|x(t) - y(t)|)$ for all $t, r \in [a, b]$ and $x, y \in \mathbb{R}$. Then the equation [\[eq12\]](#) has a unique solution $x \in C([a, b])$ for some constant λ depending on the constant s .

Proof. Let $X = C([a, b])$ and $T: X \rightarrow X$ be defined by

$$T(x)(t) = \lambda \int_a^b K(t, r, x(r)) ds, \quad \forall x \in X.$$

Let $d: X \times X \rightarrow [0, +\infty[$ be given by

$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|$$

$\forall x, y \in X$. It is clear that (X, d) is a complete b-metric space. We will find the condition on λ under which operator has a unique fixed point which will be the solution of the integral equation [\[eq12\]](#). Assume that $x, y \in X$ and $t, r \in [a, b]$. Then we get

$$\begin{aligned}
|Tx(t) - Ty(t)| &= |\lambda| \left(\left| \int_a^b K(t, r, x(r)) - \int_a^b K(t, r, y(r)) dr \right| \right) \\
&= |\lambda| \left| \int_a^b (K(t, r, x(r)) - K(t, r, y(r))) dr \right| \\
&\leq |\lambda| \int_a^b |K(t, r, x(r)) - K(t, r, y(r))| dr \\
&\leq |\lambda| \int_a^b \left(\frac{1}{s^2} |x(r) - y(r)| \right) dr \\
&= |\lambda| \frac{1}{s^2} \left[\int_a^b (|x(r) - y(r)|) dr \right].
\end{aligned}$$

This implies that

$$\begin{aligned}
\max_{t \in [a, b]} (|Tx(t) - Ty(t)|) &= \max_{t \in [a, b]} |\lambda| \int_a^b |K(t, r, x(r)) - K(t, r, y(r))| dr \\
&\leq \max_{t \in [a, b]} \frac{1}{s^2} |\lambda| \int_a^b |x(r) - y(r)| dr \\
&\leq \frac{1}{s^2} \int_a^b \left(\max_{r \in [a, b]} |x(r) - y(r)| \right) dr.
\end{aligned}$$

By the definition of b-metric space, we have $d(Tx, Ty) > 0$ and $d(x, y) > 0$ for all $x \neq y$, so we can take natural exponential sides and get

$$\begin{aligned}
e^{[s^3 d(Tx, Ty)]} &= e^{[s^3 |\lambda| \max_{t \in [a, b]} \int_a^b |K(t, r, x(r)) - K(t, r, y(r))| dr]} \\
&\leq e^{\left[\frac{|\lambda|}{s} \left(\int_a^b |x(r) - y(r)| dr \right) \right]} \\
&= \left[e^{\left[\int_a^b \max_{r \in [a, b]} |x(r) - y(r)| dr \right]} \right]^{\frac{|\lambda|}{s}},
\end{aligned}$$

provided that $|\lambda| < s$, which implies that

$$e^{[s^3 d(Tx, Ty)]} \leq e^{[s^3 \left(\int_a^b \max_{r \in [a, b]} |x(r) - y(r)| dr \right)]^k}.$$

Hence

$$F(s^3 d(Tx, Ty)) + \phi(d(x, y)) \leq F(d(x, y))$$

for all $x, y \in X$ with $\theta(t) = e^t$ and $\phi(t) = t^k$ and $k = \left(\frac{|\lambda|}{s}\right)$. It follows that T satisfies the condition [eq1] and [eq8]. Therefore there exists a unique solution of the nonlinear Fredholm inequality [eq12]. $\square$

5. Conclusion

We defined Gregus type θ and Gregus type $(\theta - \phi)$ -contraction (or R-N type contraction) on a b-metric space into itself by extending $(\theta - \phi)$ -contraction introduced Zheng et al. (Zheng) in metric space and also proved θ type theorem as well as $(\theta -$

ϕ)-type theorem in the framework of b- metric spaces. Moreover, we present an application to nonlinear integral equations. We also present illustrative example to exhibit the utility of our result.

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